M413 - Fall 2013 - HW8 - Enrique Areyan 60/60 ①

(1) Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?

Pf: Suppose that $\lim_{n\to\infty} s_n = s$. So we have that:

Given $\varepsilon > 0 : \exists N : \forall n \geq N : |s_n - s| < \varepsilon \iff -\varepsilon < s_n - s < \varepsilon \iff \begin{cases} s - \varepsilon < s_n \\ s_n < \varepsilon + s \end{cases}$

Now, we want to show that $\{|s_n|\}$ converges to $|s|$, i.e., $\lim_{n\to\infty} |s_n| = |s|$.

Let $\varepsilon > 0$. Choose N s.t. $|s_n - s| < \varepsilon$ provided that $n \geq N$. then,

$||s_n| - |s|| \leq |s_n - s|$, which can be deduced from triangular inequal

$< \varepsilon \implies ||s_n| - |s|| < \varepsilon \implies \lim_{n\to\infty} |s_n| = |s|$ //.

The converse is not true. Consider $s_n = (-1)^n$. then, $|s_n| = |(-1)^n| = 1$, so $|s_n|$ is the constant sequence 1; $|s_n| = 1, 1, 1, \ldots$ which clearly converges to

However, $s_n = (-1)^n$ does not converges since:

$$\lim_n \sup s_n = 1 \neq -1 = \lim_n \inf s_n. //$$

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(3) If $s_1 = \sqrt{2}$ and $s_{n+1} = \sqrt{2 + \sqrt{s_n}}$ $(n = 1, 2, 3, \ldots)$, prove that $\{s_n\}$ converges, an that $s_n < 2$ for $n = 1, 2, 3, \ldots$

Pf: By theorem 3.14, If we can show that $\{s_n\}$ is monotonic and bound then we will have that $\{s_n\}$ converges. So, let us prove:

(I) $\{s_n\}$ is monotonic increasing.
(II) $\{s_n\}$ is bounded by 2.

} (Both proofs by induction).

(II) BASE CASE: $s_1 = \sqrt{2} < 2$, so base case holds.

INDUCTIVE STEP: Suppose $s_n < 2$. We want to show that $s_{n+1} \overset{?}{<} 2$

By definition: $s_{n+1} = \sqrt{2 + \sqrt{s_n}}$

$< \sqrt{2 + \sqrt{2}}$ By inductive hypothesis

$< \sqrt{2 + 2}$ since $\sqrt{2} < 2$

$= \sqrt{4} = 2 \implies s_{n+1} < 2. \implies \{s_n\}$ is bounded by 2.

(I) Let us prove that $s_n < s_{n+1}$, for $n = 1, 2, 3, \ldots$, By induction:

BASE CASES: We need to prove the following base cases: $\sqrt{\ }$ is an increasing functi

$s_1 = \sqrt{2} < \sqrt{2 + \sqrt{\sqrt{2}}} = s_2$, since $\sqrt{\sqrt{2}} > 0$ and

Likewise, it is not hard to see that

$s_2 = \sqrt{2 + \sqrt{\sqrt{2}}} < \sqrt{2 + \sqrt{2 + \sqrt{\sqrt{2}}}} = s_3$. So base cases hold.

Inductive STEP: Suppose that $S_{n-1} < S_n$. We want to show that $S_n \overset{?}{<} S_{n+1}$

By inductive hypothesis we know: $S_{n-1} = \sqrt{2+\sqrt{S_{n-2}}} < \sqrt{2+\sqrt{S_{n-1}}} = S_n$. Now, by definition

$S_{n+1} = \sqrt{2+\sqrt{S_n}} \Rightarrow S_{n+1}^2 = 2+\sqrt{S_n} \Rightarrow S_{n+1}^2 - 2 = \sqrt{S_n} \Rightarrow (S_{n+1}^2-2)^2 = S_n$.

By hypothesis: $S_n = (S_{n+1}^2-2)^2 = \sqrt{2+\sqrt{S_{n-1}}} > S_{n-1} = \sqrt{2+\sqrt{S_{n-2}}}$

$\Rightarrow (S_{n+1}^2-2)^2 > \sqrt{2+\sqrt{S_{n-2}}} \Rightarrow S_{n+1}^2 - 2 > \sqrt{\sqrt{2+\sqrt{S_{n-2}}}} = \sqrt{S_{n-1}}$

$\Rightarrow S_{n+1}^2 - 2 > \sqrt{S_{n-1}} \Rightarrow S_{n+1} > \sqrt{2+\sqrt{S_{n-1}}} = S_n \Rightarrow \boxed{S_{n+1} > S_n}$ $\forall n$.

(I) & (II) $\Rightarrow \{S_n\}$ is monotonic, bounded $\Rightarrow \{S_n\}$ converges.

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5) Investigate the behavior (convergence or divergence) of Σa_n if

a) $a_n = \sqrt{n+1} - \sqrt{n}$:

Solution: $a_n = \sqrt{n+1} - \sqrt{n} = \sqrt{n+1} - \sqrt{n} \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{(\sqrt{n+1})^2 - (\sqrt{n})^2}{\sqrt{n+1}+\sqrt{n}} = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}$

$\Rightarrow a_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$. Let us try to find a bound for a_n:

$a_n = \frac{1}{\sqrt{n+1}+\sqrt{n}} > \frac{1}{\sqrt{n+1}+\sqrt{n+1}}$ since $\sqrt{n} < \sqrt{n+1}$

$= \frac{1}{2\sqrt{n+1}}$

$> \frac{1}{2(n+1)}$ since $\sqrt{n+1} < n+1$

. Now let us investigate the behavior of

b_n, where $b_n = \frac{1}{2(n+1)}$. $\Rightarrow \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2(n+1)} = \frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n}$; by change of variables.

But then Σb_n diverges by the p-test (here $p=1$).

Moreover, since $a_n > b_n > 0$ $\forall n$ & Σb_n diverges $\Big\} \Rightarrow \Sigma a_n$ diverges (theorem 3.25) (comparison Test).

b) $a_n = (\sqrt[n]{n} - 1)^n$;

By the root test: Let $\alpha = \limsup_n \sqrt[n]{|a_n|} = \lim_n |\sqrt[n]{n} - 1| = \lim_n \sqrt[n]{n} - 1 = 1 - 1 = \boxed{0}$ since $\sqrt[n]{n} \geq 1$; $\forall n \in \mathbb{N}$.

Since we proved that $\lim \sqrt[n]{n} = 1$ and that the sum of the limits is the limits of the sum. Moreover, since $\lim_n \sqrt[n]{n} - 1$ exists it must be equal to $\limsup_n \sqrt[n]{n} - 1$. Hence:

$\alpha = 0 < 1 \Rightarrow \Sigma a_n$ converges.

(11) Suppose $a_n > 0$, $S_n = a_1 + \cdots + a_n$, and $\sum_1^\infty a_n$ diverges.

(a) Prove that $\sum_1^\infty \frac{a_n}{1+a_n}$ diverges

Pf: Let us prove this by CASES:

(i) $\{a_n\}$ is bounded.
(ii) $\{a_n\}$ is not bounded.

(i) Suppose there exists $M \in \mathbb{R}$, $M > 0$ s.t. $a_n \leq M$ for all $n \in \mathbb{N}$.

then, $\sum_1^\infty \frac{a_n}{1+a_n} \geq \sum_1^\infty \frac{a_n}{1+M} = \frac{1}{1+M} \sum a_n > \sum_1^\infty a_n > 0$.

$\Rightarrow \sum_1^\infty \frac{a_n}{1+a_n} > \sum a_n > 0$; but $\sum_1^\infty a_n > 0 \Rightarrow \sum_1^\infty \frac{a_n}{1+a_n}$ diverges.

(ii) Suppose a_n is not bounded by hypothesis, $a_n > 0$, therefore it must be the case that $a_n \to \infty$ as $n \to \infty$ for otherwise a_n would be bounded. But then, since $\frac{a_n}{1+a_n} = \frac{1}{\frac{1}{a_n}+1}$, we have that

$\lim_n \frac{a_n}{1+a_n} = \lim_n \frac{1}{\frac{1}{a_n}+1} = \frac{1}{0+1} = 1$, since $\frac{1}{a_n} \to 0$ as $n \to \infty$.

$\Rightarrow \lim_n \frac{a_n}{1+a_n} = 1 \neq 0$, so the general term does not goes to zero, which implies that $\sum \frac{a_n}{1+a_n}$ diverges.

In any case (i)/(ii) $\sum \frac{a_n}{1+a_n}$ diverges.

(b) Prove that $\frac{a_{N+1}}{S_{N+1}} + \cdots + \frac{a_{N+K}}{S_{N+K}} \geq 1 - \frac{S_N}{S_{N+K}}$ and deduce that $\sum \frac{a_n}{S_n}$ diverges.

Pf: $\frac{a_{N+1}}{S_{N+1}} + \cdots + \frac{a_{N+K}}{S_{N+K}} \geq \frac{a_{N+1}}{S_{N+K}} + \cdots + \frac{a_{N+K}}{S_{N+K}}$; since $S_{n+1} > S_n$, this is because $a_n > 0$, so S_n increases at each step

$= \frac{a_{N+1} + \cdots + a_{N+K}}{S_{N+K}}$ adding fractions

$= \frac{a_{N+1} + \cdots + a_{N+K} + (a_1 + \cdots + a_N - (a_1 + \cdots + a_N))}{S_{N+K}}$ adding and substracting the same quantity

$= \frac{a_1 + \cdots + a_N + a_{N+1} + \cdots + a_{N+K} - (a_1 + \cdots + a_N)}{S_{N+K}}$ Rearranging terms

$= \frac{S_{N+K} - S_N}{S_{N+K}}$ By definition of S_n.

$= 1 - \frac{S_N}{S_{N+K}}$.

$\Rightarrow \boxed{\frac{a_{N+1}}{S_{N+1}} + \cdots + \frac{a_{N+K}}{S_{N+K}} \geq 1 - \frac{S_N}{S_{N+K}}}$ //

low we want to deduce that $\Sigma \frac{a_n}{S_n}$ diverges. Suppose to the contrary that $\Sigma \frac{a_n}{S_n}$ converges. then, by the Cauchy criterion:

Given $\varepsilon > 0$ $\exists N$: $\forall m \geq n \geq N$: $\sum_{k=n}^{m} \frac{a_k}{S_k} < \varepsilon$; (no absolute value needed since both a_k and S_k are positive).

ow, fix an integer q and let $K > q$.

hoose $\varepsilon = 1 - \frac{S_q}{S_{q+K}} > 0$ since $S_{q+K} > S_q \Rightarrow \frac{S_q}{S_{q+K}} < 1$. By the Cauchy

iterion: there exists N s.t. $\forall m \geq n \geq N$:

$$\sum_{i=n}^{m} \frac{a_i}{S_i} < \varepsilon \Leftrightarrow \frac{a_n}{S_n} + \frac{a_{n+1}}{S_{n+1}} + \dots + \frac{a_m}{S_m} < \varepsilon = 1 - \frac{S_q}{S_{q+K}};$$

t by what was proved before $\frac{a_n}{S_n} + \frac{a_{n+1}}{S_{n+1}} + \dots + \frac{a_m}{S_m} \geq 1 - \frac{S_n}{S_m}$, so we have

$$1 - \frac{S_q}{S_{q+K}} > \frac{a_n}{S_n} + \frac{a_{n+1}}{S_{n+1}} + \dots + \frac{a_m}{S_m} \geq 1 - \frac{S_n}{S_m} \Rightarrow 1 - \frac{S_q}{S_{q+K}} > 1 - \frac{S_n}{S_m}$$

$\Rightarrow \frac{S_n}{S_m} > \frac{S_q}{S_{q+K}}$. Now, q is fixed so the ratio $\frac{S_q}{S_{q+K}}$ is a fix number, Call it β.

$\Rightarrow \frac{S_n}{S_m} > \beta$. However, $S_m > S_n$; and $m \geq n \geq N$; so we can make this ratio as small as we want by letting $m \to \infty$; which contradicts The fact derived that $\frac{S_n}{S_m} > \beta$, for some fixed β.

erefore, $\Sigma \frac{a_n}{S_n}$ diverges.

Prove that $\frac{a_n}{S_n^2} \leq \frac{1}{S_{n-1}} - \frac{1}{S_n}$ and deduce that $\Sigma \frac{a_n}{S_n^2}$ converges.

:

$$\frac{1}{S_{n-1}} - \frac{1}{S_n} = \frac{S_n - S_{n-1}}{S_{n-1} S_n} \quad \text{by adding fractions}$$

$$= \frac{a_1 + \dots + a_n - (a_1 + \dots + a_{n-1})}{S_{n-1} S_n} \quad \text{by definition of } S_n$$

$$= \frac{a_n}{S_{n-1} S_n} \quad \text{Cancelling terms.}$$

$$\geq \frac{a_n}{S_n S_n} = \frac{a_n}{S_n^2} \quad \text{since } S_n \geq S_{n-1}; \text{ since } a_n > 0.$$

$$\Rightarrow \boxed{\frac{1}{S_{n-1}} - \frac{1}{S_n} \geq \frac{a_n}{S_n^2}}$$

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Now we want to deduce that $\Sigma \frac{a_n}{S_n^2}$ converges.

Let $\varepsilon > 0$. Pick N s.t. $\frac{1}{S_{N-1}} - \frac{1}{S_N} < \varepsilon$, provided that $m \geq n \geq N$. then.

$$\left|\sum_{k=n}^{m} \frac{a_k}{S_k^2}\right| = \sum_{k=n}^{m} \frac{a_k}{S_k^2} \quad \text{since all terms are positive.}$$

$$= \frac{a_n}{S_n^2} + \frac{a_{n+1}}{S_{n+1}^2} + \frac{a_{n+2}}{S_{n+2}^2} + \cdots + \frac{a_{m-1}}{S_{m-1}^2} + \frac{a_m}{S_m^2} \quad \text{Expanding sum.}$$

$$\leq \left(\frac{1}{S_{n-1}} - \cancel{\frac{1}{S_n}}\right) + \left(\cancel{\frac{1}{S_n}} - \cancel{\frac{1}{S_{n+1}}}\right) + \left(\cancel{\frac{1}{S_{n+1}}} - \cancel{\frac{1}{S_{n+2}}}\right) + \cdots + \left(\cancel{\frac{1}{S_{m-2}}} - \cancel{\frac{1}{S_{m-1}}}\right) + \left(\cancel{\frac{1}{S_{m-1}}} - \frac{1}{S_m}\right) \text{; by previous statement}$$

$$= \frac{1}{S_{n-1}} - \frac{1}{S_m} \quad \text{CANcelling terms}$$

$$\leq \frac{1}{S_{N-1}} - \frac{1}{S_N} < \varepsilon \quad \text{by our choice of } N \text{ for } \varepsilon.$$

$$\Rightarrow \left|\sum_{k=n}^{m} \frac{a_k}{S_k^2}\right| < \varepsilon \Rightarrow \Sigma \frac{a_n}{S_n^2} \text{ converges.}$$

(d) What can be said about $\Sigma \frac{a_n}{1+na_n}$ and $\Sigma \frac{a_n}{1+n^2a_n}$?

Solution: Clearly, we can compare $\Sigma \frac{a_n}{1+n^2a_n}$ wit $\Sigma \frac{1}{n^2}$ as follows:

$$n^2 + \frac{1}{a_n} > n^2 \quad \text{since } a_n > 0 \Rightarrow \frac{1}{a_n} > 0. \quad (n = 1, 2, 3, \ldots)$$

$$\frac{1+n^2a_n}{a_n} > n^2 \quad \text{rewritting } n^2 + \frac{1}{a_n}$$

taking the reciprocal in both sides

$$0 < \frac{a_n}{1+n^2a_n} < \frac{1}{n^2} \quad \text{; summing both sides of last expres}$$

$$\Rightarrow \Sigma \frac{1}{n^2} > \Sigma \frac{a_n}{1+n^2a_n} > 0$$

theorem 3.28 (p-test) $\Rightarrow \Sigma \frac{1}{n^2}$. Now, theorem 3.25 (compariso test) $\Rightarrow \Sigma \frac{a_n}{1+n^2a_n}$ converges.

Now, $\Sigma \frac{a_n}{1+na_n}$ may converge or diverge. Consider:

① $a_n = \frac{1}{n}$. this satisfies our hypothesis, i.e., $a_n > 0$, Σa_n d

then, $$\frac{a_n}{1+na_n} = \frac{\frac{1}{n}}{1+\cancel{n}\frac{1}{\cancel{n}}} = \frac{\frac{1}{n}}{2} = \frac{1}{2n} > \frac{1}{n} > 0 \text{ ; since } \Sigma \frac{1}{n} \text{ divers}$$

by TH 3.25 (comparison test) $\Rightarrow \frac{a_n}{1+na_n}$ diverges. 20

(II) Let $a_n = n^2$. Clearly, $\sum a_n$ diverges. and $a_n > 0$. But

$$\frac{a_n}{1+na_n} = \frac{n^2}{1+nn^2} = \frac{n^2}{1+n^3} = \frac{1}{3}\left[\frac{1}{1+n} + \frac{2n-1}{n^2-n+1}\right] < \frac{2}{n^2} \text{; and}$$

since $\sum \frac{1}{n^2}$ converges, so does $\frac{a_n}{1+na_n}$.

Therefore, $\frac{a_n}{1+na_n}$ might diverge by (I) or it might converge by (II).

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6) Fix a positive number α. Choose $x_1 > \sqrt{\alpha}$, and define $x_2, x_3, x_4, \dots$ by the recursion formula: $x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right)$.

) Prove that $\{x_n\}$ decreases monotonically and that $\lim x_n = \sqrt{\alpha}$. $\forall n$.

f: First, let us show that x_n is bounded below by $\sqrt{\alpha}$, i.e., $x_n > \sqrt{\alpha}$ by induction:

BASE CASE: $n=1$, then by hypothesis $x_1 > \sqrt{\alpha}$.

Inductive STEP: Suppose that $x_n > \sqrt{\alpha}$. WANT TO SHOW $x_{n+1} \overset{?}{>} \sqrt{\alpha}$ (Note that x_n is never zero).

Since $x_n > \sqrt{\alpha} \Rightarrow x_n^2 > \alpha \Rightarrow x_n > \frac{\alpha}{x_n}$. Now consider:

$\Rightarrow \sqrt{x_n} > \sqrt{\frac{\alpha}{x_n}} = \frac{\sqrt{\alpha}}{\sqrt{x_n}} \Rightarrow \sqrt{x_n} - \frac{\sqrt{\alpha}}{\sqrt{x_n}} > 0$

$\left(\sqrt{x_n} - \frac{\sqrt{\alpha}}{\sqrt{x_n}}\right)^2 > 0 \Rightarrow x_n - 2\sqrt{\alpha} + \frac{\alpha}{x_n} > 0 \Rightarrow x_n + \frac{\alpha}{x_n} > 2\sqrt{\alpha}$

$\Rightarrow \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) > \sqrt{\alpha} \Rightarrow \boxed{x_{n+1} > \sqrt{\alpha}}$ //

Second, let us show that x_n is monotonically decreasing, i.e., $x_n \geq x_{n+1}$. This follows as a corollary to what we proved before because we know that for any n: $x_n > \sqrt{\alpha}$. Therefore,

$\Rightarrow x_n^2 > \alpha \Rightarrow 0 > \alpha - x_n^2 \Rightarrow 0 > \alpha + x_n^2 - 2x_n^2 \Rightarrow 2x_n^2 > \alpha + x_n^2$

$\Rightarrow 2x_n > \frac{\alpha + x_n^2}{x_n}$ (again, x is alway positive)

$\Rightarrow x_n > \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) = x_{n+1} \Rightarrow x_n > x_{n+1}$, so x is strictly monotonic decreasing.

By theorem 3.14, since x_n is monotonic bounded it follows that x_n converges. Let $\beta = \lim_{n\to\infty} x_n$.

Now we want to prove that $\lim_{n\to\infty} x_n \stackrel{?}{=} \sqrt{\alpha}$.

$$\beta = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{2}\left(x_{n-1} + \frac{\alpha}{x_{n-1}}\right) = \frac{1}{2}\left[\lim_{n\to\infty} x_{n-1} + \lim_{n\to\infty} \frac{\alpha}{x_{n-1}}\right]$$

$$= \frac{1}{2}\left[\beta + \frac{\alpha}{\beta}\right] = \beta \Rightarrow 2\beta = \beta + \frac{\alpha}{\beta} \Rightarrow \beta = \frac{\alpha}{\beta} \Rightarrow \beta^2 = \alpha \Rightarrow \boxed{\beta = \sqrt{\alpha}}$$

Since $x_n > 0$, we care only about $+\sqrt{\alpha}$ and disregard $-\sqrt{\alpha}$.

(b) Put $\varepsilon_n = x_n - \sqrt{\alpha}$, and show that: $\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$

Pf: $\varepsilon_{n+1} = x_{n+1} - \sqrt{\alpha} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - \sqrt{\alpha}$ by def. of x_n

$$= \frac{x_n}{2} + \frac{\alpha}{2x_n} - \sqrt{\alpha}$$

$$= \frac{x_n^2 + \alpha - 2\sqrt{\alpha}x_n}{2x_n} \quad \text{adding fractions}$$

$$= \frac{(x_n - \sqrt{\alpha})^2}{2x_n} = \frac{\varepsilon_n^2}{2x_n} \quad \text{by definition of } \varepsilon_n$$

Now, since we proved that $x_n > \sqrt{\alpha}\ \forall n$, we get

$$\boxed{\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}}$$

, so that setting $\beta = 2\sqrt{\alpha}$ we get:

$$\varepsilon_{n+1} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}} = \frac{\varepsilon_n^2}{\beta} < \frac{\left(\frac{\varepsilon_{n-1}^2}{\beta}\right)^2}{\beta} = \frac{\frac{\varepsilon_{n-1}^4}{\beta^2}}{\beta} = \frac{\varepsilon_{n-1}^{2^2}}{\beta\cdot\beta^2} = \beta\left(\frac{\varepsilon_{n-1}}{\beta}\right)^{2^2} < \cdots < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n}$$

(c) If $\alpha = 3$ and $x_1 = 2$, show that $\varepsilon_1/\beta < \frac{1}{10}$.

By definitions: $\frac{\varepsilon_1}{\beta} = \frac{x_1 - \sqrt{\alpha}}{2\sqrt{\alpha}} = \frac{2-\sqrt{3}}{2\sqrt{3}} = \frac{2-\sqrt{3}}{2\sqrt{3}} \cdot \frac{2+\sqrt{3}}{2+\sqrt{3}} = \frac{4-3}{2\sqrt{3}(2+\sqrt{3})} = \frac{1}{2\sqrt{3}(2+\sqrt{3})}$

And since $2\sqrt{3}(2+\sqrt{3}) = 4\sqrt{3} + 2\cdot 3 = 4\sqrt{3} + 6 > 4 + 6 > 10$ (since $\sqrt{3} > 1$)

$$\Rightarrow \frac{1}{2\sqrt{3}(2+\sqrt{3})} < \frac{1}{10} \Rightarrow \frac{\varepsilon_1}{\beta} < \frac{1}{10}$$

Finally, we can conclude $\longrightarrow$ (back page)

$\varepsilon_5 = x_5 - \sqrt{\alpha}$; $\varepsilon_6 = x_6 - \sqrt{\alpha}$. We proved:

$\varepsilon_{n+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n}$ Apply it here:

$$\varepsilon_{4+1} = \varepsilon_5 < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^4} = 2\sqrt{3}\left(\frac{\varepsilon_1}{\beta}\right)^{16} < 2\sqrt{3}\left(\frac{1}{10}\right)^{16} = 2\sqrt{3}\cdot 10^{-16} < 4\cdot 10^{-16}$$

$$\Rightarrow \boxed{\varepsilon_5 < 4\cdot 10^{-16}}$$. Likewise:

$$\varepsilon_{5+1} = \varepsilon_6 < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^5} = 2\sqrt{3}\left(\frac{\varepsilon_1}{\beta}\right)^{32} < 2\sqrt{3}\left(\frac{1}{10}\right)^{32} = 2\sqrt{3}\cdot 10^{-32} < 4\cdot 10^{-32}$$

$$\Rightarrow \boxed{\varepsilon_6 < 4\cdot 10^{-32}}$$

So,

ε_n measures the error between the n^{th} term in $\{x_n\}$ and $\sqrt{\alpha}$
The above calculations show that the error reduces by an order of 10^{-16} when computing succesive terms x_4, x_5. This means that we approach very quickly $\Theta(2^n)$ to the value of $\sqrt{\alpha}$.
Hence, this is an algorithm with exponential convergence.

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